Necessary and Sufficient Conditions for Uniqueness in Spatial Equilibria: The Case of Two Locations

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February 25, 2019

Abstract

In this paper we provide a necessary and sufficient condition for equilibrium uniqueness in a two-location version of the economic geography model in Allen and Arkolakis (2014). Compared to the well-know sufficient condition provided by Allen and Arkolakis, our condition for uniqueness allows for positive agglomeration externalities even in the absence of congestion effects, and highlights the key role played by two additional parameters: the trade elasticity, which regulates the strength of the dispersion force associated with the decline in the terms of trade caused by migration into a location; and trade costs, which weaken this dispersion force by limiting trade across locations.

*We thank Cecile Gaubert for very useful conversations. All errors are our own.
1. Introduction

Recently there has been a surge of research on quantitative models of the spatial distribution of economic activity that are tractable yet sophisticated enough to capture first-order features of the data, such as heterogeneous geography, productivity, and amenities, along with trade and migration across locations. Given that economies of agglomeration are a key force in economic geography, a fundamental issue for the application of these models entails characterizing conditions under which the equilibrium is well behaved.

Allen and Arkolakis (2014, henceforth AA), provide sufficient conditions for uniqueness of spatial equilibria for a generalized framework that nests an important class of economic geography models. The condition hinges on the balance between two key parameters: one which governs the strength of local production externalities ($\psi$), a force for the concentration of economic activity; and another which regulates the strength of congestion externalities associated with the crowding of local amenities ($\delta$), a force for the dispersion of economic activity. If there are positive production externalities and negative congestion externalities, then the condition requires that the latter is weakly stronger than the former, $\psi + \delta \leq 0$. This is a strong condition that, for example, rules out agglomeration externalities if there are no congestion externalities. Moreover, since the condition is not necessary, multiplicity of equilibria need not arise if it is violated.

In this paper, we provide necessary and sufficient conditions for uniqueness in a two-region version of the AA model. We find that, in general, the condition requires that trade costs be low enough. More specifically, denoting $\tau \geq 1$ as the symmetric iceberg trade cost and $\epsilon$ as the trade elasticity, the condition is that $\tau < \tau^*$, where $\tau^*$ is a function of parameters $\epsilon$, $\psi$ and $\delta$, as well as the relative exogenous productivity and amenity differences across locations.

For the special case in which locations are symmetric, we derive an explicit formula

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1Nested models include, for example, Krugman (1991), Helpman (1998) and Redding (2016).
2Kucheryavyy, Lyn and Rodríguez-Clare (2018) provide necessary and sufficient conditions for uniqueness of equilibria for a generalized framework that nests an important class of multi-industry trade models, including ones that allow for agglomeration externalities in production. Their condition is that the product of the agglomeration (scale) and trade elasticities is weakly lower than one, $\psi \epsilon \leq 1$. 
for $\tau^*$,
\[
\tau^* \equiv \left( \frac{2 + \psi - \delta}{(\psi + \delta)(1 + 2\epsilon)} \right)^{1/\epsilon}.
\]  
(1)

Not surprisingly, the upper bound on trade costs is decreasing in $\psi$ and increasing in $\delta$. More importantly, here we now see the role played by the trade elasticity, $\epsilon$, which is absent in AA’s sufficiency condition $\psi + \delta \leq 0$. The trade elasticity matters for uniqueness because it regulates the strength of terms of trade changes associated with changes in population across locations – a force for the dispersion of economic activity. In particular, terms of trade worsen for a location experiencing a rise in population, and this effect is more severe with a lower trade elasticity. This implies that a lower $\epsilon$ increases the range of trade costs for which the equilibrium is unique.

The role of the trade elasticity is most easily seen in the case with frictionless trade and no congestion, where $\psi < 1/\epsilon$ is both necessary and sufficient for uniqueness. This inequality implies that the dispersion force associated with terms of trade dominates agglomeration effects and hence uniqueness is guaranteed even in the absence of congestion effects.

The importance of the terms-of-trade dispersion force also explains why, if $0 < \psi + \delta < (1 - \delta)/\epsilon$, then uniqueness holds if and only if trade costs are low enough, $\tau < \tau^* < \infty$. The reason is simple: trade costs weaken this dispersion force by limiting trade, thereby making terms of trade changes less relevant. As a result, with high trade costs, it is attractive for consumers to concentrate in larger locations made attractive by agglomeration externalities that dominate congestion effects.

In the next section, we outline the general theoretical framework. In Section 3 we state and prove necessary and sufficient conditions for uniqueness in a two-location version of the model. In Section 4, we offer some concluding remarks.

2. Spatial Equilibrium Model

Consider the multi-region spatial model as in AA. Formally, there are $N$ locations indexed by $i$, $l$ and $n$, and each location produces a unique differentiated variety of a good. Trade costs are of the standard iceberg type, so that delivering a unit of a good from location $i$ to location $n$ requires shipping $\tau_{ni} \geq 1$ units of the good, with $\tau_{ii} = 1$ for all $i$ and $\tau_{nl} \leq \tau_{ni}\tau_{li}$ for all $n$, $l$, $i$ and $k$ (triangular inequality). There are $L$ workers
who can move freely across locations and derive utility from a local amenity and the
consumption of a constant-elasticity-of-substitution (CES) aggregate of differentiated
varieties with elasticity of substitution $\sigma$. Letting $w_i$ denote equilibrium nominal wage
and $P_i$ the CES price index, the welfare of individuals residing in location $i$ is given by

$$U_i \equiv \frac{w_i}{P_i} u_i,$$

with the term $u_i$ denoting the amenity in location $i$ given by

$$u_i \equiv \bar{u}_i L_i^\delta,$$

where $\bar{u}_i$ is the exogenous utility of amenity in $i$, $L_i$ is the number of workers residing
in location $i$, and $\delta$ governs the strength of congestion externalities affecting utility.

Labor is the only factor of production, and is inelastically supplied by workers in
the location in which they reside, for which they are compensated with wage $w_i$. Each
worker in location $i$ produces $A_i$ units of a good, with the local productivity given by

$$A_i \equiv \bar{A}_i L_i^\psi,$$

where $\bar{A}_i$ is the exogenous component of productivity in location $i$, and $\psi$ regulates the
strength of agglomeration externalities affecting production.

The equilibrium conditions are that: (i) local goods markets clear; (ii) welfare is
equalized across all inhabited locations; and (iii) aggregate labor markets clear. Form-
ally, the set of equilibrium conditions is given by

$$w_i L_i = \sum_{n=1}^{N} \lambda_{ni} w_n L_n,$$  \hspace{1cm} (4)

$$L_i \geq 0, \quad \bar{U} - U_i \geq 0, \quad L_i (\bar{U} - U_i) = 0,$$  \hspace{1cm} (5)

$$\sum_{i=1}^{N} L_i = \bar{L},$$  \hspace{1cm} (6)

where $\bar{U} > 0$ is a constant,

$$\lambda_{ni} \equiv \bar{A}_i L_i^\alpha (w_i \tau_{ni})^{-\epsilon} P_n^\epsilon$$
denotes the share of expenditure that region \( n \) devotes to imports from region \( i \),

\[
P_n = \left[ \sum_{j=1}^{N} \tilde{A}_j^* L_j^* \left( w_j \tau_{nj} \right)^{-\epsilon} \right]^{-\frac{1}{\epsilon}},
\]

(7)

\( \epsilon \equiv \sigma - 1 \) is the trade elasticity, and \( \alpha \equiv \epsilon \psi \).

Following AA, we call an equilibrium \textit{regular}, if all locations are inhabited (\( L_i > 0 \) for all \( i \)), and we call an equilibrium \textit{irregular}, if some of the locations are uninhabited.

3. The Case with Two Locations

In this Section, we provide necessary and sufficient conditions for uniqueness in the case with two locations and symmetric trade costs, \( \tau_{12} = \tau_{21} = \tau \).

**Proposition 1.** Assume that \( \psi \geq 0, \delta \leq 0 \).

(i) If \( \psi + \delta \leq 0 \), then there is a unique equilibrium, and this equilibrium is regular.

(ii) If \( 0 < \psi + \delta < (1 - \delta)/\epsilon \), then any equilibrium is regular, and there exists \( \tau^* > 1 \) such that the equilibrium is unique if \( \tau < \tau^* \), whereas there are three equilibria if \( \tau > \tau^* \).

If, in addition to that, \( (\tilde{A}_2/\tilde{A}_1)^{1+\psi} (\tilde{u}_2/\tilde{u}_1)^{1-\delta} = 1 \), then

\[
\tau^* = \left( \frac{2 + \psi - \delta}{(\psi + \delta)(1 + 2\epsilon)} \right)^{1/\epsilon}
\]

and the equilibrium is unique for \( \tau = \tau^* \); otherwise, there are two equilibria for \( \tau = \tau^* \).

(iii) If \( \psi + \delta = (1 - \delta)/\epsilon \), then there are two irregular equilibria: one with \( L_1 = 0 \) and one with \( L_2 = 0 \). Additionally, if \( 1/\tau < (\tilde{u}_2/\tilde{u}_1)^{1-\delta} (\tilde{A}_2/\tilde{A}_1)^{\frac{\epsilon}{1+\epsilon}} \) < \( \tau \), there is one regular equilibrium; otherwise, there are no regular equilibria.

(iv) If \( \psi + \delta > (1 - \delta)/\epsilon \), then there are two irregular equilibria and one regular equilibrium.

Before proceeding to the proof, it is instructive to make two comments. First, condition \( \psi + \delta < (1 - \delta)/\epsilon \) is equivalent to condition \( 1 - \alpha - \delta (1 + \epsilon) > 0 \), which, as is proven by AA, guarantees that all equilibria are regular. When this condition is violated, the economy has all possible irregular equilibria. Since this is already established by AA
for the general case of $N$ locations, we refer the reader to AA’s paper for the proof of existence of irregular equilibria. In the proof of Proposition 1 below we focus only on regular equilibria.

Second, AA’s sufficient condition for uniqueness is that $\psi + \delta \leq 0$. This is a strong condition that, for example, rules out agglomeration externalities if there are no congestion externalities. In Proposition 1 we characterize a necessary and sufficient condition for uniqueness — case (ii) in the proposition — that allows for strictly positive agglomeration externalities even in the absence of congestion effects.

Let us now turn to the proof of Proposition 1. We do it by a series of lemmas. We start with the following lemma:

**Lemma 1.** In the case of regular equilibria, analysis of multiplicity of solutions of the equilibrium system (4)-(6) is equivalent to the analysis of the multiplicity of positive solutions of the following equation in $y$:

$$
\phi a^\kappa b^{1-\kappa} y^{(\psi+\delta)\epsilon-\eta} + aby^{(\psi+\delta)\epsilon} = 1 + \phi a^{1-\kappa} b^\kappa y^\eta,
$$

where $\kappa \equiv \frac{1+\epsilon}{1+2\epsilon}$, $\eta \equiv \frac{1+\psi+(\psi+\delta)\epsilon}{2+1\epsilon}$, $a \equiv (\tilde{u}_2/\tilde{u}_1)^\epsilon$, $b \equiv (\tilde{A}_2/\tilde{A}_1)^\epsilon$, and $\phi \equiv \tau^{-\epsilon}$ denotes the trade freeness parameter.

**Proof.** In the case of regular equilibria, the complementary slackness condition (5) implies that both locations have the same welfare $\tilde{U}$, and we can combine (2) and (3) for the two locations and get

$$
\left( \frac{L_2}{L_1} \right) ^ \delta = \left( \frac{w_2/w_1}{P_2/P_1} \frac{\tilde{u}_2}{\tilde{u}_1} \right) ^{-1}.
$$

Let $y \equiv L_2/L_1$. Also, by choice of numeraire set $w_1 = 1$ and $w_2 = w$. Dividing onto each other price indices for the two countries given by expression (7), we can find

$$
P_1/P_2 = \left[ \frac{1 + by^a \phi w^{-\epsilon}}{\phi + by^a w^{-\epsilon}} \right] ^{-\frac{1}{\gamma}}.
$$

After substituting this expression into (9) and doing some algebra, we get

$$
aw^{\epsilon} (\phi + by^a w^{-\epsilon}) = (1 + \phi by^a w^{-\epsilon}) y^{\delta\epsilon}.
$$

(10)
Next, we can write the goods market clearing condition (4) for the first location simply as \( 1 = \lambda_{11} + \lambda_{21} wy \), with

\[
\lambda_{11} = \frac{1}{1 + by^a \phi w^{-\epsilon}} \quad \text{and} \quad \lambda_{21} = \frac{\phi}{\phi + by^a w^{-\epsilon}}.
\]

Then, after substituting the above expressions for \( \lambda_{11} \) and \( \lambda_{21} \) into the goods market clearing condition for the first location, and after doing simple algebra, we get

\[
b(\phi + by^a w^{-\epsilon}) = (1 + \phi by^a w^{-\epsilon}) w^{1+\epsilon} y^{1-\alpha}. \tag{11}
\]

Dividing (10) by (11) and solving for \( w \), gives

\[
w = a^{-\frac{1}{1+\epsilon}} b^{\frac{1}{1+\epsilon}} y^{\frac{\alpha-\delta-1}{1+\epsilon}}. \tag{12}
\]

Substituting \( w \) from above into either (10) or (11), we get equation (8).

Equilibria of the original system (4)-(6) are then characterized by solutions \( y > 0 \) of equation (8). Any solution \( y > 0 \) of (8) can be substituted into (12) to get the equilibrium wage. Also, combining solutions of (8) with the labor market clearing condition (6) will give equilibrium labor allocations.

Lemma 1 implies that we can focus on characterizing solutions of equation (8) with the purpose of analyzing multiplicity of equilibria of the original system (4)-(6).

**Lemma 2.** If \( \psi + \delta = 0 \) or trade is free (i.e., \( \phi = 1 \)), then equation (8) has a unique positive solution.

**Proof.** In both cases with \( \psi + \delta = 0 \) and with free trade we can find solutions explicitly. If \( \psi + \delta = 0 \), then we necessarily have that \( \eta \neq 0 \) (more precisely, \( \eta > 0 \)), and there is only one positive solution to (8),

\[
y = \left( \frac{(ab - 1) + \sqrt{(ab - 1)^2 + 4\phi^2 ab}}{2\phi a^{1-\kappa} b^\kappa} \right)^{\frac{1}{\eta}}.
\]

Similarly, in the case of free trade, the only positive solution to (8) is

\[
y = \left( a^\kappa b^{1-\kappa} \right)^{\frac{1}{\eta(\psi + \delta)}}.
\]
In light of Lemma 2, for the rest of the proof of Proposition 1 we assume that \(\psi + \delta \neq 0\) and \(\phi < 1\). To characterize solutions of equation (8), let us introduce the change of variables \(r = y(\psi + \delta)c\) and work with the transformed equation

\[
V(r) \equiv \phi a^\kappa b^{1-\kappa} r^{1-\mu} - \phi a^{1-\kappa} b^\kappa r\mu + abr - 1 = 0,
\]

where \(\mu \equiv \frac{\eta}{(\psi + \delta)c}\). Qualitatively, we have four different cases: \(\mu < 0\), \(0 \leq \mu < 1\), \(\mu = 1\), and \(\mu > 1\). We analyze each case below in a separate lemma.

**Lemma 3.** If \(\mu < 0\), then equation (13) has a unique solution.

**Proof.** In case of \(\mu < 0\), it is instructive to write equation (13) as

\[
\phi a^\kappa b^{1-\kappa} r^{1-\mu} + abr = \phi a^{1-\kappa} b^\kappa r\mu + 1.
\]

The left-hand side of this equation is an increasing function that ranges from 0 to \(+\infty\) as \(r\) ranges from 0 to \(+\infty\), while the right-hand side of this equation is a decreasing function that ranges from \(+\infty\) to 1 as \(r\) ranges from 0 to \(+\infty\). Hence, the functions on the left-hand side and the right-hand side intersect exactly one time for some positive \(r\). Thus, equation (13) has a unique solution for \(\mu < 0\).

**Lemma 4.** If \(0 \leq \mu < 1\), then equation (13) has a unique solution.

**Proof.** In case of \(0 \leq \mu < 1\), it is instructive to write equation (13) as

\[
a^\kappa b^{1-\kappa} r^{1-\mu} = \frac{\phi a^{1-\kappa} b^\kappa r\mu + 1}{a^{1-\kappa} b^\kappa r\mu + \phi}.
\]

If \(\mu = 0\), then this equation gives us an explicit solution for \(r\), which is positive. In case of \(0 < \mu < 1\), the left-hand side of this equation is an increasing function that ranges from 0 to \(+\infty\) as \(r\) ranges from 0 to \(+\infty\). The derivative of the right-hand side of the above equation is

\[
\frac{(\phi^2 - 1) \mu a^{1-\kappa} b^\kappa r^{\mu-1}}{(a^{1-\kappa} b^\kappa r^\mu + \phi)^2},
\]
which is negative because $\phi < 1$. Hence, the right-hand side of the above equation is a decreasing function. Then, after noting that this function ranges from $1/\phi$ to 1 as $r$ ranges from 0 to $+\infty$, we can conclude that the left-hand side and the right-hand side of the above equation intersect exactly once at some $r > 0$. Thus, equation (13) has a unique solution for $0 \leq \mu < 1$. □

**Lemma 5.** Suppose that $\mu = 1$. If $\phi < a^\kappa b^{1-\kappa} < 1/\phi$, then equation (13) has a unique positive solution, otherwise equation (13) does not have positive solutions.

**Proof.** In case of $\mu = 1$, we can solve equation (13) explicitly,

$$r = \frac{1 - \phi a^\kappa b^{1-\kappa}}{a b - \phi a^{1-\kappa} b^\kappa}.$$

The above expression is positive if and only if we have $\phi < a^\kappa b^{1-\kappa} < 1/\phi$. If the latter condition is not satisfied, equation (13) does not have a positive solution. □

**Lemma 6.** Suppose that $\mu > 1$. There exists $\phi^* \in (0, 1)$ such that if $\phi \in (\phi^*, 1)$ then equation (13) has a unique solution, while if $\phi \in (0, \phi^*)$ then equation (13) has three solutions. In addition, if $a^{\mu+1-1} b^{\mu-\kappa} = 1$, then equation (13) has a unique solution for $\phi = \phi^*$ and $\phi^* = \frac{1}{2\mu-1}$, otherwise, equation (13) has two solutions for $\phi = \phi^*$.

**Proof.** To prove this lemma, we need to consider separately two cases: $a^{\mu+1-1} b^{\mu-\kappa} \leq 1$ and $a^{\mu+1-1} b^{\mu-\kappa} > 1$. The role of these inequalities will become clear in the process of proving the lemma.

**The case with $a^{\mu+1-1} b^{\mu-\kappa} \leq 1$.** It is immediate to see that in the case of $\mu > 1$ we have $\lim_{r \to 0} V(r) = +\infty$ and $\lim_{r \to +\infty} V(r) = -\infty$, where the second limit follows from the fact that the second term of $V$ dominates the third (linear) term when $\mu > 1$. So, $V$ has at least one zero for $r \in (0, +\infty)$. Consider the first and second derivatives of $V$,

$$V'(r) = \phi(1-\mu)a^\kappa b^{1-\kappa} r^{-\mu} - \phi a^{1-\kappa} b^\kappa r^{\mu-1} + ab,$$

$$V''(r) = -\phi(\mu-1) a^{1-\kappa} b^\kappa r^{\mu-1} - \phi a^\kappa b^{1-\kappa} r^{-\mu} + ab.$$

In the case of $\mu > 1$, we have that $\lim_{r \to 0} V'(r) = \lim_{r \to +\infty} V'(r) = -\infty$, and $V''(r) > 0$ for $r < r_0 \equiv (a/b)^{2\mu-1}$ and $V''(r) < 0$ for $r > r_0$. Hence, $V'(r)$ achieves its maximum at $r_0$. Then, depending on the sign of $V'(r_0)$, the original function $V(r)$ can be decreasing...
for all \( r \neq r_0 \), or \( V(r) \) can have alternating regions of where it increases and where it decreases.

Simple algebra reveals that

\[
V'(r_0) = ab\left(1 - \phi/\tilde{\phi}\right),
\]

where \( \tilde{\phi} \equiv \frac{1}{2^\mu - 1} \left[a\mu + \kappa - 1 b\mu - \kappa \right]^{2^\mu - 1} \). Since we consider the case with \( a\mu + \kappa - 1 b\mu - \kappa \leq 1 \), we have that \( \tilde{\phi} < 1 \). Then, if \( \phi = \tilde{\phi} \), we have that \( V'(r_0) = 0 \) and \( V'(r) < 0 \) for all \( r \neq r_0 \), while if \( \phi > \tilde{\phi} \), we have that \( V'(r) < 0 \) for all \( r > 0 \). Hence, overall, in the case of \( \phi \geq \tilde{\phi} \) there is a unique solution. This case is illustrated in Figure 1 by the curves labeled “\( \phi = \tilde{\phi} \)” and “\( \phi > \tilde{\phi} \)”.

As one can guess after looking at Figure 1, condition \( \phi \geq \tilde{\phi} \) is only a sufficient condition for uniqueness. We now proceed to show that there exists a \( \phi^* \in (0, \tilde{\phi}] \) such that for \( \phi > \phi^* \) there is a unique solution, while for \( \phi < \phi^* \) there are three solutions. The case with \( \phi = \phi^* \) is the threshold case in which we can have a unique solution if \( \phi^* = \tilde{\phi} \) or two solutions if \( \phi^* < \tilde{\phi} \).

For any \( \phi < \tilde{\phi} \) we have that \( V'(r_0) > 0 \). Then, since \( \lim_{r \to -\infty} V'(r) = \lim_{r \to +\infty} V'(r) = -\infty \), and \( V'(r) \) increases for \( r < r_0 \) and decreases for \( r > r_0 \), we have that in the case of \( \phi < \tilde{\phi} \) there are exactly two values \( r_1 \) and \( r_2 \) with \( 0 < r_1 < r_0 < r_2 \), such that \( V'(r_1) = V'(r_2) = 0 \). This fact allows us to define functions \( r_1(\phi) \equiv \{ r | r \leq r_0 \text{ and } V'(r) = 0 \} \) and \( r_2(\phi) \equiv \{ r | r \geq r_0 \text{ and } V'(r) = 0 \} \) both with domain \( \phi \in (0, \tilde{\phi}) \). We have that \( V'(r) < 0 \) for \( r \in (0, r_1(\phi)) \cup (r_2(\phi), +\infty) \), while \( V'(r) > 0 \) for \( r \in (r_1(\phi), r_2(\phi)) \) (on Figure 1 this case is illustrated by the curves labeled “\( \phi < \phi^* \)”).

This implies that \( V(r) \) is decreasing for \( r \in (0, r_1(\phi)) \), increasing for \( r \in (r_1(\phi), r_2(\phi)) \), and decreasing again for \( r \in (r_2(\phi), +\infty) \). Hence, \( r_1(\phi) \) is a local minimum of \( V(r) \), while \( r_2(\phi) \) is a local maximum of \( V(r) \). Moreover,
since \( V'(r) > 0 \) for \( r \in (r_1(\phi), r_2(\phi)) \), we have that \( V(r_1(\phi)) < V(r_0) < V(r_2(\phi)) \).

Next, evaluation of \( V(r_0) \) for any \( \phi \) reveals that

\[
V(r_0) = a^{\frac{2(\mu+\kappa-1)}{2\mu-1}} b^{\frac{2(\mu-\kappa)}{2\mu-1}} - 1.
\]

Observe that \( V(r_0) \) does not depend on \( \phi \). Since we consider the case with \( a^{\mu+\kappa-1} b^{\mu-\kappa} \leq 1 \), we have that \( V(r_0) \leq 0 \). Then for any \( \phi < \tilde{\phi} \) we have that \( V(r_1(\phi)) < 0 \). Fix any \( \phi < \tilde{\phi} \) and suppose that we also have \( V(r_2(\phi)) < 0 \). Then, since \( \lim_{r \to -\infty} V(r) = +\infty \) and \( V(r) \) is decreasing for \( r < r_1(\phi) \), we have that \( V(r) = 0 \) for exactly one \( r \in (0, r_1(\phi)) \). At the same time, despite the fact that \( V(r) \) is increasing on the interval \((r_1(\phi), r_2(\phi))\), it never crosses the horizontal axis, because \( V(r_2(\phi)) < 0 \). Combined with the fact that \( V(r) \) is decreasing for \( r > r_2 \), this means that \( V(r) \) never crosses the horizontal axis for \( r > r_1 \). Thus, having \( V(r_2(\phi)) < 0 \) implies that we have a unique solution.

Now suppose that for some fixed \( \phi < \tilde{\phi} \) we have \( V(r_2(\phi)) > 0 \). Again, \( V(r) = 0 \) for exactly one \( r \in (0, r_1(\phi)) \). Also, since \( \lim_{r \to +\infty} V(r) = -\infty \) and \( V(r) \) is decreasing for \( r > r_2 \), we have that \( V(r) = 0 \) for exactly one \( r > r_2(\phi) \). Finally, since \( V(r) \) is increasing on the interval \((r_1(\phi), r_2(\phi))\), we have that \( V(r) = 0 \) for exactly one \( r \in (r_1(\phi), r_2(\phi)) \). Therefore, having \( V(r_2(\phi)) > 0 \) implies that we have three solutions.

Consider the case with \( a^{\mu+\kappa-1} b^{\mu-\kappa} = 1 \). In this case \( V(r_0) = 0 \). Hence, \( r_0 \) is a solution of equation \( V(r) = 0 \) for any \( \phi \). And since we have already established that equation \( V(r) = 0 \) has a unique solution for \( \phi \geq \tilde{\phi} \), we can conclude that this unique solution is \( r_0 \). For any \( \phi < \tilde{\phi} \), we have that \( V(r_1(\phi)) < 0 < V(r_2(\phi)) \). Hence, as we have argued above, in this case there are three solutions to equation \( V(r) = 0 \). Overall, we conclude that in the case of \( a^{\mu+\kappa-1} b^{\mu-\kappa} = 1 \) we have \( \phi^* = \tilde{\phi} \) such that \( V(r) = 0 \) has a unique solution for \( \phi \geq \phi^* \) and three solutions for \( \phi < \phi^* \).

Now consider the case with \( a^{\mu+\kappa-1} b^{\mu-\kappa} < 1 \). Let us show that there exists a \( \phi^* < \tilde{\phi} \) such that \( V(r_2(\phi^*)) = 0 \). By its definition, \( r_2(\phi) \) solves equation \( V'(r) = 0 \). The other solution of this equation is \( r_1(\phi) \). Since \( V(r_1(\phi)) < 0 \) for all \( \phi < \tilde{\phi} \), then \( r_2(\phi) \) is the only solution to \( V'(r) = 0 \) that can also simultaneously be a solution to \( V(r) = 0 \). Therefore, to show existence of \( \phi^* \) such that \( V(r_2(\phi^*)) = 0 \), we can just forget for a moment about the function \( r_2(\phi) \) and consider the system of two equations \( V'(r) = 0 \) and \( V(r) = 0 \) in two unknowns \( r \) and \( \phi \). Any solution to this system — call it \((r^*, \phi^*)\) — will automatically
satisfy the condition \( \phi^* < \tilde{\phi} \) (because in the case of \( a^{\mu+\kappa-1} b^{\mu-\kappa} < 1 \) equation \( V'(r) = 0 \) has a solution in \( r \) only under this condition) and will give \( r_2(\phi^*) = r^* \).

Equation \( V'(r) = 0 \) implies

\[
\phi = \frac{1}{(\mu - 1) a^{\kappa - 1} b^{-\kappa} r^{-\mu} + \mu a^{-\kappa} b^{\kappa - 1} r^{\mu - 1}}.
\]  

(14)

Observe that \( \phi > 0 \) for any \( r \). Substituting \( \phi \) from the above expression into \( V(r) = 0 \) gives the equation that determines \( r^* \):

\[
H(r) \equiv (\mu - 1) a^{2-2\kappa} b^{2\kappa} r^{2\mu} - \mu a^{1-2\kappa} b^{2\kappa - 1} r^{2\mu - 1} + \mu abr - (\mu - 1) = 0.
\]

We have that \( \lim_{r \to 0} H(r) = -(\mu - 1) < 0 \), while \( \lim_{r \to +\infty} H(r) = +\infty \). Hence, there is at least one \( r^* > 0 \) that solves \( H(r) = 0 \) with the corresponding \( \phi^* > 0 \) given by expression (14).

Now consider any \( \phi^* < \tilde{\phi} \) such that \( V(r_2(\phi^*)) = 0 \). Let us show that \( V(r_2(\phi)) < 0 \) for \( \phi \in (\phi^*, \tilde{\phi}) \), while \( V(r_2(\phi)) > 0 \) for \( \phi \in (0, \phi^*) \). Recall that for any \( \phi < \tilde{\phi} \) we have that \( r_2(\phi) \) is a local maximum of \( V(r) \). Formally, \( r_2(\phi) \) solves the problem \( \max_r \{V(r) | r \geq r_0\} \). Therefore, we can apply the envelope theorem and obtain

\[
\frac{\partial V(r_2(\phi))}{\partial \phi} = a^{1-\kappa} b^{\kappa} r_2(\phi)^{1-\mu} \left( r_0^{2\mu-1} - r_2(\phi)^{2\mu-1} \right).
\]

Since \( r_2(\phi) > r_0 \), we have that \( \partial V(r_2(\phi)) / \partial \phi < 0 \), that is, \( V(r_2(\phi)) \) is a (strictly) decreasing function of \( \phi \) for \( \phi \in (0, \tilde{\phi}) \). Therefore, we necessarily have that \( V(r_2(\phi)) < 0 \) for \( \phi \in (\phi^*, \tilde{\phi}) \), while \( V(r_2(\phi)) > 0 \) for \( \phi \in (0, \phi^*) \). This also establishes the uniqueness of \( \phi^* < \tilde{\phi} \) such that \( V(r_2(\phi^*)) = 0 \) and proves that equation \( V(r) = 0 \) has a unique solution for \( \phi \in (\phi^*, \tilde{\phi}) \) and three solutions for \( V(r_2(\phi)) > 0 \) for \( \phi \in (0, \phi^*) \). The case with \( \phi = \phi^* \) is the threshold case in which the horizontal axis in the \((r, V)\) coordinates is tangent to \( V \) at \( r_2(\phi^*) \), and there are two solutions to equation \( V(r) = 0 \) (on Figure 1 this case is illustrated by the curves labeled “\( \phi = \phi^* \)”).

Since earlier we have established that \( V(r) = 0 \) has a unique solution for \( \phi \geq \tilde{\phi} \), we conclude that \( V(r) = 0 \) has a unique solution for all \( \phi > \phi^* \). This completes the proof of the current lemma for the case with \( a^{\mu+\kappa-1} b^{\mu-\kappa} \leq 1 \).
The case with $a^{\mu+\kappa-1} b^{\mu-\kappa} > 1$. We can write $V(r) = -abr \tilde{V}(1/r)$, where

$$
\tilde{V}(r) \equiv \phi \tilde{a}^\kappa \tilde{b}^{1-\kappa} r^{1-\mu} - \phi \tilde{a}^{1-\kappa} \tilde{b}^\kappa r^\mu + \tilde{a} \tilde{b} r - 1,
$$

with $\tilde{a} \equiv a^{-1}$ and $\tilde{b} \equiv b^{-1}$. Obviously, $V(r)$ and $\tilde{V}(r)$ have the same number of solutions. Observe that $\tilde{V}(r)$ is almost identical to $V(r)$ with the only difference that parameters $a$ and $b$ are relabeled to $\tilde{a}$ and $\tilde{b}$. Moreover, we now have $\tilde{a}^{\mu+\kappa-1} \tilde{b}^{\mu-\kappa} < 1$. Hence, we can just apply the analysis above to function $\tilde{V}(r)$ and conclude that there exists a $\phi^* \in (0, \tilde{\phi})$ with $\tilde{\phi} \equiv \frac{1}{2^{\mu-1}} \{ \tilde{a}^{\mu+\kappa-1} \tilde{b}^{\mu-\kappa} \}^{\frac{1}{2^{\mu-1}}} < 1$ such that the equation $\tilde{V}(r) = 0$ has a unique solution for $\phi > \phi^*$, it has two solutions for $\phi = \phi^*$, and it has three solutions for $\phi \in (0, \phi^*)$. This completes the proof of the lemma.

\begin{lemma}
The following equivalences hold

\begin{enumerate}
    \item[(i)] $\mu < 0 \iff -(1+\psi)/\epsilon < \psi + \delta < 0$;
    \item[(ii)] $0 \leq \mu < 1 \iff (\psi + \delta \leq -(1+\psi)/\epsilon \text{ or } \psi + \delta > (1-\delta)/\epsilon)$;
    \item[(iii)] $\mu = 1 \iff \psi + \delta = (1-\delta)/\epsilon$;
    \item[(iv)] $\mu > 1 \iff 0 < \psi + \delta < (1-\delta)/\epsilon$;
\end{enumerate}

\end{lemma}

\begin{proof}
The proof of this lemma is a straightforward analysis of the cases for $\mu$ and subcases for $\psi + \delta$:

\begin{enumerate}
    \item[(i)] $\mu < 0$: If $\psi + \delta > 0$, then

    $$
    \mu < 0 \iff \frac{1 + \psi + (\psi + \delta) \epsilon}{(\psi + \delta)(2\epsilon + 1)} < 0 \iff \psi + \delta < -(1+\psi)/\epsilon,
    $$

    which is inconsistent. So, in the case of $\mu < 0$, we must have $\psi + \delta < 0$. In this case $\mu < 0$ is equivalent to $\psi + \delta > -(1+\psi)/\epsilon$. Hence, overall, $\mu < 0$ is equivalent to $-(1+\psi)/\epsilon < \psi + \delta < 0$

    \item[(ii)] $0 \leq \mu < 1$: If $\psi + \delta > 0$, then $\mu \geq 0$ is equivalent to $\psi + \delta \geq -(1+\psi)/\epsilon$, which holds for $\psi + \delta > 0$. Also, we have

    $$
    \mu < 1 \iff \frac{1 + \psi + (\psi + \delta) \epsilon}{(\psi + \delta)(2\epsilon + 1)} < 1 \iff \psi + \delta > (1-\delta)/\epsilon.
    $$

    So, the case $0 \leq \mu < 1$ is consistent with inequality $\psi + \delta > (1-\delta)/\epsilon$.

    Now consider the possibility of $\psi + \delta < 0$. In this case we have that $\mu \geq 0$ is equivalent to $\psi + \delta \leq -(1+\psi)/\epsilon$, and $\mu < 1$ is equivalent to $\psi + \delta < (1-\delta)/\epsilon$. Therefore, the case
$0 \leq \mu < 1$ is also consistent with inequality $\psi + \delta \leq - (1 + \psi) / \varepsilon$.

Thus, overall, we get that $0 \leq \mu < 1$ is equivalent to having either $\psi + \delta > (1 - \delta) / \varepsilon$ or $\psi + \delta \leq - (1 + \psi) / \varepsilon$.

(iii) $\mu = 1$: In this case we immediately find that $\psi + \delta = (1 - \delta) / \varepsilon$.

(iv) $\mu > 1$: If $\psi + \delta > 0$, then $\mu > 1$ is equivalent to $\psi + \delta < (1 - \delta) / \varepsilon$. Hence, the case with $\mu > 1$ is consistent with $0 < \psi + \delta < (1 - \delta) / \varepsilon$.

Now consider the possibility of $\psi + \delta < 0$. In this case $\mu > 1$ is equivalent to $\psi + \delta > (1 - \delta) / \varepsilon$, which is inconsistent, because $\delta < 0$.

Thus, overall, we get that $\mu > 1$ is equivalent to $0 < \psi + \delta < (1 - \delta) / \varepsilon$. \hfill $\square$

We can now combine the results from all lemmas to get the proof of all the parts of Proposition 1 for the case of regular equilibria. Combining the results of Lemma 7 parts (i) and (ii) with the results of Lemmas 2, 3, and 4, we get the result of Proposition 1 part (i). The result of Proposition 1 part (ii) is obtained by combining the result of Lemma 7 part (iv) with the result of Lemma 6. The result of Proposition 1 part (iii) is obtained by combining the result of Lemma 7 part (iii) with the result of Lemma 5. Finally, the result of Proposition 1 part (iv) is obtained by combining the result of Lemma 7 part (ii) with the result of Lemma 4. This completes the proof of Proposition 1.

3.1. Discussion

We conclude by noting that while generally the upper bound on trade costs is implicit, there are two instructive cases for which it is an explicit function of the model parameters. The first is the case with symmetric locations, which we have already discussed extensively in the introduction. Here we simply remind the reader that for this case we can explicitly see that the upper bound on trade costs is decreasing in $\psi$ and increasing in $\delta$, a feature which holds more generally. More importantly, for this case the key role played by the trade elasticity $\varepsilon$ – a dispersion force for economic activity – in shaping uniqueness becomes transparent.

The second case involves a sufficient condition which allows us to see explicitly the role of asymmetry in the attractiveness across locations as measured by a weighted composite of the relative (exogenous) productivity and amenity differences across locations, $T \equiv a^{\mu + \kappa - 1} b^{\mu - \kappa}$. Consider the case in Lemma 6 in which agglomeration exter-
nalities dominate congestion effects and by extension imply that both \( \mu + \kappa - 1 \) and \( \mu - \kappa \) are strictly positive. As noted in Lemma 6, it is without loss of generality to consider the case in which \( T \leq 1 \), with a smaller \( T \) implying a higher degree of asymmetry in attractiveness across locations. A sufficient condition for uniqueness is

\[
\tau \leq \tilde{\tau} \equiv T^{-\frac{1}{2(2\mu - 1)}} \left[ \frac{2 + \psi - \delta}{(\psi + \delta)(1 + 2\epsilon)} \right]^{1/\epsilon},
\]

(15)

where \( \tilde{\tau} < \tau^* \). Noting that \( 2\mu - 1 > 0 \), this implies that higher asymmetry relaxes the upper-bound on trade costs for equilibrium uniqueness.

To get a sense of potential magnitudes for which uniqueness holds in the absence of congestion, we use parameter values \( \epsilon = 5 \) as in Head and Mayer (2014) and Costinot and Rodríguez-Clare (2014), and \( \psi = 0.1 \) as in AA. For the symmetric case these values imply an upper-bound on trade costs \( \tau^* = 1.14 \). Without symmetry, equation (15) together the fact that \( \tilde{\tau} < \tau^* \) imply that if one location is twice as attractive as the other then the upper bound on trade costs increases to \( \tau^* > \tilde{\tau} = 1.23 \).

4. Concluding Remarks

In this paper, we provide a necessary and sufficient condition for uniqueness in a two-location version of a generalized economic-geography model. The condition requires that trade costs be low enough, with the upper-bound on trade costs becoming more stringent with higher agglomeration externalities and more relaxed with greater congestion effects. In contrast to the previous well known sufficient condition, our condition allows that for positive agglomeration externalities even in the absence of congestion externalities. This is possible thanks to the role played by the dispersion force associated with a finite trade elasticity, which implies that a location's terms of trade worsen as it gets larger. Since the importance of trade magnifies the strength of this force, trade costs combined with the trade elasticity now play an important role in ensuring uniqueness.

The main limitation of our analysis, of course, is that it is restricted to two locations. We see our results as suggestive of the importance of trade costs and the trade elasticity in determining whether the equilibrium is unique in more general environments, but providing a tight condition there seems quite challenging. We conjecture that in the
case with \( N > 2 \) symmetric locations then the equilibrium is unique if \( \tau < \tau^* \), with \( \tau^* \) defined as in Proposition 1.

**References**


